

Rainbow Graphs and Switching Classes

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Abstract

A rainbow graph is a graph that admits a vertex-coloring such that every color appears exactly once in the neighborhood of each vertex. We investigate some properties of rainbow graphs. In particular, we show that there is a bijection between the isomorphism classes of n -rainbow graphs on $2n$ vertices and the switching classes of graphs on n vertices.

1 Introduction

A vertex coloring of a graph (without loops and multiple edges) is called an *n -rainbow coloring* if every color appears once, and only once, in each neighborhood of a vertex. A graph is called an *n -rainbow* graph if the graph admits a rainbow coloring. Note that an n -rainbow coloring is not a proper coloring. We state here some general facts about rainbow graphs, which are all easy to verify.

1. An n -rainbow graph is n -regular.
2. Each color occurs equally often in the vertex set of an n -rainbow graph.
3. An n -rainbow graph contains a perfect matching, which is constructed by choosing edges whose endpoints have the same color.
4. The number of vertices of an n -rainbow graph is a multiple of $2n$.

On the other hand, *switching* a vertex v in a labeled graph G (without loops and multiple edges) means reversing the adjacency of v and w for all other vertices w . That is, if v and w were connected by an edge, we delete the edge. If they were not, we add that edge. This operation was defined by Seidel [4]. Two labeled graphs are *switching equivalent* if one can be obtained from the other by a sequence of switching operations. Two unlabeled graphs are switching equivalent if there are labeled graphs corresponding to them which are switching equivalent. Such equivalent classes are called *switching classes*. Mallows and Sloane [3] and Cameron [1] showed that the number of switching classes equals to the number of unlabelled even graphs.

The main result of this paper is the following theorem.

Theorem 1.1. *The number of isomorphism classes of n -rainbow graphs with $2n$ vertices is the same as the number of switching classes of graphs with n vertices.*

Corollary 1.2. *The number of isomorphism classes of n -rainbow graphs with $2n$ vertices is the same as the number of unlabelled even graphs (every vertex has every degree) with n vertices.*

We will explicitly construct a bijective map between the classes.

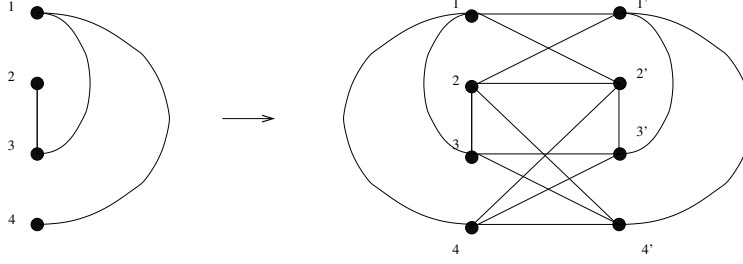


Figure 1: The map that sends G to \tilde{G} .

2 Map from switching classes to n -rainbow graphs

In this section, we introduce a map from switching classes of graphs having n vertices onto isomorphism classes of n -rainbow graphs having $2n$ vertices. This map will be shown to be a bijection in Section 3.

Choose a graph $G = (V, E)$ that has n vertices without loops and multiple edges. Label the vertices from 1 to n . Now let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a graph having $2n$ vertices such that:

1. the vertices are labeled by $\{1, \dots, n, 1', \dots, n'\}$,
2. $(i, i') \in \tilde{E}$.
3. $(i, j), (i', j') \in \tilde{E}$ if $(i, j) \in E$.
4. $(i', j), (i, j') \in \tilde{E}$ if $(i, j) \notin E$.

Then the underlying unlabeled graph of \tilde{G} is n -rainbow because we can color the vertices labeled by i and i' with the color i , and this is clearly a rainbow coloring. An example is given in Figure 1.

We denote by $[G]$ the switching class of G . Now let us define the desired map ψ by sending $[G]$ to the underlying unlabeled graph of \tilde{G} . To see that ψ is well-defined, let H be the graph obtained from G by switching at a vertex labeled by i . Let \tilde{H} be the graph with $2n$ vertices obtained from H by the previous method. Now if we interchange the labels of i, i' in \tilde{H} , we recover \tilde{G} . An example of this phenomenon is given in Figure 2.

Proposition 2.1. ψ is surjective.

Proof. For any n -rainbow graph \tilde{G} on $2n$ vertices, assign an n -rainbow coloring on its vertex set. Now let G be an induced subgraph of \tilde{G} having n vertices on which each color appears once. Clearly, $\psi([G]) = \tilde{G}$. \square

We remark here that a similar map appears in [2], with a different coloring of graphs, in an attempt to show that the switching equivalence problem is polynomial time equivalent to the graph isomorphism problem.

3 The main proof

In this section, we will prove our main result, Theorem 1.1. To do so, let us recall the definition of Seidel matrices, originally defined in [4].

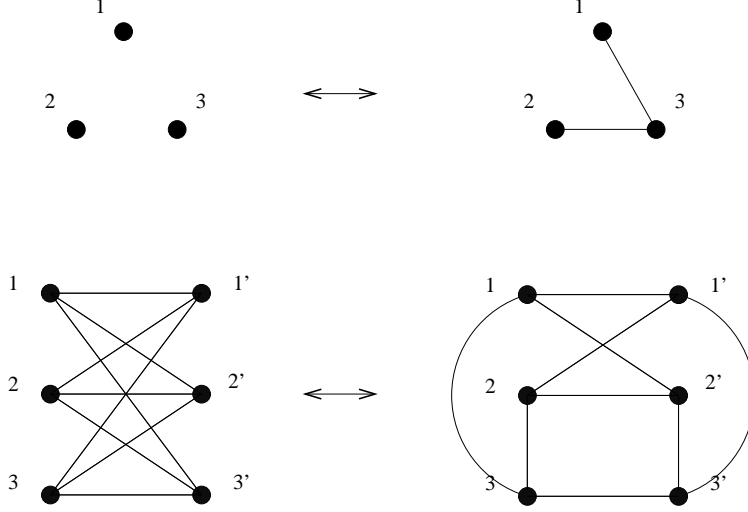


Figure 2: The exchange.

Definition 3.1. Let G be a graph over n vertices. The *Seidel matrix* of G is a symmetric $n \times n$ matrix with entries a_{ij} such that:

- $a_{ii} = 0$ for all $i \in [n]$,
- $a_{ij} = a_{ji} = 1$ if $(i, j) \in E(G)$,
- $a_{ij} = a_{ji} = -1$ if $(i, j) \notin E(G)$.

Definition 3.2. For a Seidel matrix A we define \tilde{A} as

$$\tilde{A} := \begin{pmatrix} A & I - A \\ I - A & A \end{pmatrix}.$$

Note that \tilde{A} is also a Seidel matrix. In particular, if A is the Seidel matrix of G , then \tilde{A} is the Seidel matrix of \tilde{G} .

Let A and B be Seidel matrices of some graphs G and G' , respectively. Then G and G' are isomorphic if and only if $B = PAP^{-1}$ for some permutation matrix P . Furthermore, G and G' are switching equivalent if and only if $B = QAQ^{-1}$ for some signed permutation matrix Q .

In order to prove Theorem 1.1, it only remains to prove that ψ is injective, since surjectivity is given by Proposition 2.1. We need to show that if \tilde{G} is isomorphic to \tilde{G}' , then G and G' are in the same switching class. This can be restated in the following linear algebra language:

Proposition 3.3. *If two Seidel matrices A and B satisfy $\tilde{A} = P\tilde{B}P^{-1}$ where P is a $2n \times 2n$ permutation matrix, then there exists an $n \times n$ signed permutation matrix Q such that $A = QBQ^{-1}$.*

We prove this proposition by a sequence of lemmas given below.

Lemma 3.4. *Let $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$ be a $2n \times 2n$ permutation matrix, where P_1, \dots, P_4 are $n \times n$ matrices. The following equations hold:*

$$1) P_1 P_3^T = 0, P_2 P_4^T = 0, P_3 P_1^T = 0, P_4 P_2^T = 0.$$

$$2) P_1 P_1^T + P_2 P_2^T = I, P_3 P_3^T + P_4 P_4^T = I.$$

Proof. From $PP^T = I$ we have

$$\begin{pmatrix} P_1 P_1^T + P_2 P_2^T & P_1 P_3^T + P_2 P_4^T \\ P_3 P_1^T + P_4 P_2^T & P_3 P_3^T + P_4 P_4^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The claim follows from the fact that every entry of each of P_i 's is nonnegative. \square

Lemma 3.5. *In the assumption of Proposition 3.3 and Lemma 3.4, let Z denote $(P_1 - P_2 - P_3 + P_4)/2$. Then, we have*

$$A - I = Z(B - I)Z^T.$$

Proof. $\tilde{A} = P\tilde{B}P^{-1}$ implies

$$\begin{pmatrix} A & I - A \\ I - A & A \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} B & I - B \\ I - B & B \end{pmatrix} \begin{pmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{pmatrix}.$$

It follows from Lemma 3.4 that:

$$A = XBX^T + I - XX^T,$$

$$A = XBY^T + I - XY^T,$$

$$A = YBY^T + I - YY^T,$$

$$A = YBX^T + I - YX^T,$$

where $X = P_1 - P_2$ and $Y = P_4 - P_3$. Summing these four equations, we get:

$$4A = (X + Y)B(X + Y)^T + 4I - (X + Y)(X + Y)^T.$$

Hence,

$$A - I = \left(\frac{X + Y}{2} \right) (B - I) \left(\frac{X + Y}{2} \right)^T.$$

\square

Definition 3.6. Let S be an $m \times m$ matrix. We call S a **signed half-permutation** matrix if every row and column of S has either

- 1) exactly one nonzero entry α with $\alpha = \pm 1$, or
- 2) exactly two nonzero entries α and β with $\alpha = \pm \frac{1}{2}$, $\beta = \pm \frac{1}{2}$.

For instance, a matrix

$$\begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

is a signed half-permutation matrix.

Definition 3.7. Let S be an $m \times m$ signed half-permutation matrix. We say that an $m \times m$ matrix T is an *integration* of S if the following conditions hold for all $i = 1, \dots, m$.

- 1) If the i -th row(resp. column) of S has only one nonzero entry(which is ± 1), then the i -th row(resp. column) of T is the same as the i -th row(resp. column) of S .
- 2) If the i -th row(resp. column) of S has two nonzero enties, so that it is of form $[\dots \frac{(-1)^p}{2} \dots \frac{(-1)^q}{2} \dots]$, then the i -th row(resp. column) of T is obtained by doubling one of the nonzero entries and setting the other entry to zero. In other words, the i -th row(resp. column) of T should look like either $[\dots (-1)^p \dots 0 \dots]$ or $[\dots 0 \dots (-1)^q \dots]$.

Clearly an integration of a signed half-permutation matrix is a signed *permutation* matrix. For instance, the following matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an integration of the following signed half-permutation matrix

$$\begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Lemma 3.8. *Every signed half-permutation matrix has an integration.*

Proof. We can find an integration T of the signed half-permutation matrix S by the following method:

First, pick an entry α_1 of S with value $\alpha_1 = \pm \frac{1}{2}$ in S . Replace α_1 by $\text{sgn}(\alpha_1)$, where $\text{sgn}(\alpha)$ is 1 if α is positive and is -1 if α is negative. Let $\alpha_2 = \pm \frac{1}{2}$ be the other nonzero entry in the same row as α_1 . Replace α_2 by 0. Let $\alpha_3 = \pm \frac{1}{2}$ be the other nonzero entry in the same column as α_2 . Replace α_3 by $\text{sgn}(\alpha_3)$. By continuing this process, alternating row and column, we will end up at α_1 again since there are only finite number of entries. If there is no entry with value $\pm \frac{1}{2}$ then we are done. Otherwise, we can repeat this process.

It is straightforward that this algorithm will give an integration of a signed half-permutation matrix S . \square

Lemma 3.9. *Let M, N be $m \times m$ matrices such that each entry m_{ij} of M satisfies $-1 \leq m_{ij} \leq 1$ and every entry of N is either 1 or -1 . If $MZ = N$ for some signed half-permutation matrix Z , then every entry of M is either 1 or -1 . Furthermore, if Q is an integration of Z then we have $MQ = N$.*

Proof. If j -th column of Z has an entry $z_{ij} = \pm 1$, then $m_{ki} = \pm n_{kj} = \pm 1$ for all $k = 1, \dots, m$. Otherwise, it has two nonzero entries, $z_{i_1 j} = \pm \frac{1}{2}$ and $z_{i_2 j} = \pm \frac{1}{2}$ for some i_1 and i_2 . In this case, $-1 \leq m_{ki_1} z_{i_1 j} + m_{ki_2} z_{i_2 j} \leq 1$, and the equalities are satisfied only if $m_{ki_1} = 2n_{kj} z_{i_1}$ and $m_{ki_2} = 2n_{kj} z_{i_2}$. Hence every entry of M is either 1 or -1 . Moreover, if Q is an integration of Z , then it's a signed permutation matrix with $q_{ij} = z_{ij}$ if $z_{ij} = \pm 1$ and $q_{ij} = 2z_{ij}$ or 0 if $z_{ij} = \pm \frac{1}{2}$. Therefore $m_{ki} q_{ij} = n_{kj}$, and we have $MQ = N$. \square

Now we are ready to prove Proposition 3.3, which would finish off the proof of Theorem 1.1.

Proof of Proposition 3.3. By Lemma 3.5, $A - I = Z(B - I)Z^T$, where $Z = (P_1 - P_2 - P_3 + P_4)/2$. We can see that Z is a signed half-permutation matrix. Let Q be an integration of Z . Obviously Q^T is an integration of Z^T . Since A, B are Seidel matrices, every entry of $A - I$ is either 1 or -1 , and every entry of $Z(B - I)$ is between -1 and 1. By Lemma 3.9, every entry of $Z(B - I)$ is either 1 or -1 , and $A - I = Z(B - I)Q^T$. Using Lemma 3.9 again on the transpose of $Z(B - I)$, we get $Z(B - I) = Q(B - I)$. Therefore $A - I = Q(B - I)Q^T = QBQ^T - I$, hence $A = QBQ^T$. \square

Acknowledgment We would like to thank Tanya Khovanova for introducing a puzzle that led us to the study of rainbow graphs: A sultan decides to check how wise his two wise men are. The sultan chooses a cell on a chessboard and shows it only to the first wise man. In addition, each cell on the chessboard either contains a pebble or is empty. Then, the first wise man either removes a pebble from a cell that contains a pebble or adds a pebble to an empty cell. Next, the second wise man must look at the board and guess which cell was chosen by the sultan. The two wise men are permitted to agree on the strategy beforehand. The strategy for ensuring that the second wise man will always guess the chosen cell comes from assigning a rainbow coloring on a 64-regular graph.

References

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